

Signature Change on the Brane

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We explore the possibility of having a good description of classical signature change in the brane scenario.

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The aim of this letter is to show, in simple terms, that a natural scenario for the change of signature in the physical spacetime is provided by the brane-world models [1–3] (see also [4–6] for an exhaustive list of references) or, in general, by every higher-dimensional theory [7] which may contain domain walls and/or branes.

The main idea behind our proposal is that d -branes are nothing but *timelike* ($d+1$)-surfaces in a higher-dimensional spacetime (the bulk) [8]. However, nothing prevents the possibility of having *perfectly regular* branes which change its character from (say) spacelike to timelike, or which are partly null, or even more complicated possibilities. The first case corresponds to a signature-changing brane. The interesting property is that both the bulk and the brane can be regular everywhere even though the change of signature may appear as a dramatical event when seen from within the brane. Notice that the signature in the bulk is left unchanged, so that our work differs significantly from other recent studies [9]. In our proposal, the study of the change of signature becomes the simple geometrical analysis of imbedded submanifolds in the bulk: a well-posed mathematical problem without pathologies. It is remarkable that many of the traditional *ad hoc assumptions* concerning signature change [10] are shown to become pure *necessary conditions* in the brane case, which indirectly proves the plausibility of our idea and makes it worth exploring it, possibly shedding some light into the “signature-change controversy” [10].

Whether a signature change occurred in our effective spacetime is debatable, and several independent works have considered this possibility [12,11]. From a classical viewpoint, a signature change may serve to avoid the singularities of general relativity [13], such as the big-bang, which might be replaced by a Euclidean region prior to the birth of time. Signature change has also been vindicated as an effective classical description of both the no-boundary proposal [14] and the quantum tunneling [15] approach for the prescription of the Universe’s wave function in quantum cosmology. In general, every process which can be studied by resorting to the “imaginary time”, e.g. [14], can be also analyzed by means of change of the signature. All these possibilities could be naturally considered in our proposal.

As a matter of concreteness we will focus on the re-

cent models based on a single 3-brane embedded into a five-dimensional Lorentzian manifold with a noncompact fifth dimension [3] (a more geometrically focused review of this model can be found in [5]) and [16,17]. We can think of such a brane model as consisting of two Lorenztian regions joined together across corresponding smooth timelike boundaries. The matching between the two manifolds can be performed as long as the induced metrics on the two boundaries are isometric. The spacetime thus built is the bulk and the joining hypersurface Σ is the brane. The energy-momentum tensor on the brane can be calculated directly from the discontinuity [K_{ab}] of the second fundamental forms of Σ by using Israel’s formula [18]. Other kind of models with a compact fifth dimension, or based on two branes (e.g. [2]), or with a brane as the boundary of a single spacetime [4], could be treated analogously.

In order to describe signature-changing branes we only need to relax the condition that Σ is timelike, but then the usual matching conditions are no longer valid (in particular, the Israel formula) and the appropriate generalization must be used. Fortunately, this generalization was already developed in [19] in general relativity. The results carry over to any dimension with no essential change and can therefore be used to study signature-changing branes. So, let (\mathcal{V}, g) be a 5-dimensional spacetime and Σ a smooth hypersurface $\Sigma \subset \mathcal{V}$. The causal character of Σ is allowed to change along the hypersurface. More precisely, we assume that Σ contains three regions Σ^E , S and Σ^L where the hypersurface is spacelike, null, and timelike respectively: Σ^E will correspond to the Euclidean phase of the brane and Σ^L to the Lorentzian one, while the set S (assumed, for definiteness, to have empty interior in Σ) is the signature-changing set. The brane Σ has a well-defined smooth normal 1-form n_μ , which is timelike on Σ^E , spacelike on Σ^L and becomes null at S . The induced metric on Σ , or first fundamental form h_{ab} , is correspondingly positive-definite at Σ^E , Lorentzian at Σ^L , and degenerate at S [19]. As is known, one degeneration vector at S is precisely \vec{n} (index upstairs), which is tangent to Σ at S [19]. A simple but important consequence of the construction is the following result [20].

Result I.1 *The signature-changing set S is a smooth spacelike three-dimensional surface. The induced metric*

h_{ab} of Σ has $\vec{n}|_S$ as unique degeneration direction at S .

In plain words, by choosing the reference system appropriately, this means that the signature change takes place at an *instant of time*. Since this property is desirable, it has always been implicitly *assumed*. However, in a pure 4-dimensional spacetime, not imbedded in a bulk, there exist *many* other possibilities. Interestingly, this becomes now a prediction, providing a first clear example of how the brane scenario can lead to strong limitations on the allowed possibilities thereby proving that the traditional ad hoc assumptions are justified and natural.

But, can we actually produce a sensible signature-changing brane? To answer this, we have examined the traditional ways of building explicit branes. The simplest, and most frequently used, method to construct them is to cut a spacetime across a timelike hypersurface and join it to an identical copy of itself across the boundary. The resulting bulk has a Z_2 -symmetry with respect to the brane. It is natural to ask whether a similar construction can produce signature-changing branes.

Result I.2 [20] *It is impossible to join two identical copies of a spacetime with signature-changing boundary Σ , across Σ , to produce a bulk with continuous metric.*

Hence, the Z_2 mirror symmetry is incompatible with a signature-changing brane. Therefore, for signature-changing branes, more sophisticated constructions are necessary, such as gluing two different regions of the same spacetime, or two different spacetimes across appropriate hypersurfaces. Another consequence is that this result may select the proper construction for the Riemann tensor of a manifold with boundary [4], because the Z_2 -symmetry used in one of the two procedures in [4] cannot be invoked when the boundary changes its character.

Another standard procedure is the use of umbilical hypersurfaces, i.e. those for which the second fundamental form is proportional to the first one everywhere. This immediately implies that the energy-momentum tensor on the brane is of cosmological constant type. However

Result I.3 [20] *A smooth umbilical hypersurface must have constant signature. Moreover, if $[K_{ab}] = Fh_{ab} \neq 0$ on a brane Σ , then its signature must remain constant.*

Therefore, everywhere umbilical branes cannot undergo a change of signature. A physical consequence is that signature-changing branes cannot have a Λ -term energy-momentum tensor *everywhere*. Hence, some fields must become excited at least near the signature-changing set S . This seems to indicate the existence of some dynamical quantum processes for the fields present, responsible for the eventual change of signature. Nevertheless, our treatment is intended to describe a pure classical limit of any quantum mechanism leading to the signature change,

and it has enough freedom to allow for specific models in this direction.

The above Results show essential differences between signature-changing and standard timelike branes. In the sequel, we show the existence of signature-changing branes with several desirable features by presenting an explicit example [20].

Because of its importance in the Randall-Sundrum models [2,3] and as is customary in brane and string works, the bulk will be taken to be anti-de Sitter space-time, AdS_5 , which in adequate coordinates has the metric

$$ds^2 = -(1 + \lambda^2 \rho^2)dt^2 + (1 + \lambda^2 \rho^2)^{-1}d\rho^2 + \rho^2 d\Omega_{S^3}^2, \quad (1)$$

where $d\Omega_{S^3}^2$ is the round metric of S^3 , $\rho > 0$, $\lambda > 0$ is a constant. The cosmological constant is $\Lambda = -6\lambda^2$ (flat bulk as $\lambda \rightarrow 0$). For the sake of simplicity we will only consider the spherically symmetric hypersurfaces Σ described by $F(t, \rho) = 0$, or equivalently in parametric form (ignoring the angular part) by $t(\xi)$, $\rho(\xi)$, where ξ is the parameter. With this assumption, the first fundamental form of Σ is

$$ds^2|_\Sigma = N(\xi)d\xi^2 + a^2(\xi)d\Omega_{S^3}^2, \quad (2)$$

where $a(\xi) \equiv \rho(\xi)$, $N \equiv -n_\mu n^\mu = (1 + \lambda^2 a^2)^{-1}\dot{a}^2 - (1 + \lambda^2 a^2)t^2$ and overdot means $d/d\xi$. The change of signature corresponds obviously to a change in the sign of $N(\xi)$. Expression (2) has two desirable features: the Lorentzian part of Σ describes a standard (closed) Robertson-Walker (RW) cosmological model; and the change of signature happens everywhere at some instant of cosmological time, thus replacing the universal big bang. The model still has one free function of ξ which can be chosen according the particular situation being tackled.

We can now proceed to the construction of the brane. Due to Result I.2, we cannot use the standard procedure of gluing two copies of AdS_5 across the boundary Σ . However, we can still keep AdS_5 as our global bulk by taking *another different* AdS_5 with a *different* cosmological constant $\tilde{\Lambda} = -6\tilde{\lambda}^2$ and line-element

$$d\tilde{s}^2 = -(1 + \tilde{\lambda}^2 \tilde{\rho}^2)d\tilde{t}^2 + (1 + \tilde{\lambda}^2 \tilde{\rho}^2)^{-1}d\tilde{\rho}^2 + \tilde{\rho}^2 d\Omega_{S^3}^2, \quad (3)$$

and a new spherically symmetric hypersurface $\tilde{\Sigma}$ given in parametric form by $\tilde{\rho}(\xi)$ and $\tilde{t}(\xi)$. A necessary requisite in order to build a well-defined bulk by pasting Σ with $\tilde{\Sigma}$ is that the corresponding first fundamental forms (2) of Σ in AdS_5 and of $\tilde{\Sigma}$ in \tilde{AdS}_5 be isometric. This fixes $\tilde{\Sigma}$ completely (except for isometries) as the solution of $\tilde{\rho}(\xi) = \rho(\xi) \equiv a(\xi)$ and of the differential equation

$$\dot{\tilde{t}}^2 = (1 + \tilde{\lambda}^2 a^2)^{-2}[\dot{a}^2 - N(\xi)(1 + \tilde{\lambda}^2 a^2)]. \quad (4)$$

By using the results in [19] we can compute the energy-momentum tensor of the resulting bulk which, as in the standard timelike case, has a distributional part $T_{\mu\nu}|_\Sigma = \delta \cdot \tau_{\mu\nu}$, where δ is a typical scalar distribution

with support on the brane. Some care is needed here, because the definition of δ requires a choice of volume form [19], which is canonical when the hypersurface is everywhere non-null, but not for a signature-changing Σ . Nevertheless, $\delta \cdot \tau_{\mu\nu}$ is independent of this choice [19] (but $\tau_{\mu\nu}$ is choice dependent!). Selecting the volume 4-form $\eta = a^3 d\xi \wedge \eta_{S^3}$ on Σ , with η_{S^3} the standard measure in S^3 , and for a matching as in figure 1 (where $t > 0$ along Σ [20]) the explicit expression for $\tau_{\mu\nu} dx^\mu dx^\nu$ reads

$$-\frac{\varrho}{N} [\dot{t}(1 + \lambda^2 a^2) dt - \dot{a}(1 + \lambda^2 a^2)^{-1} d\rho]^2 + p a^2 d\Omega_{S^3}^2, \quad (5)$$

$$\varrho = \frac{3}{a\kappa_5^2} [\sqrt{\dot{a}^2 - N(1 + \lambda^2)} - \sqrt{\dot{a}^2 - N(1 + \tilde{\lambda}^2)}] \quad (6)$$

where κ_5 is the coupling constant corresponding to the Einstein equations in 5 dimensions, and p can be obtained from the following conservation equation

$$\dot{\varrho} - \varrho \frac{\dot{N}}{2N} + \frac{3\dot{a}}{a} (\varrho + p) = 0. \quad (7)$$

A simple analysis shows that $\tau_{\mu\nu}$ and $\delta \cdot \tau_{\mu\nu}$ are regular everywhere on Σ . Equation (7) takes the usual RW form for the variables $\varrho |N|^{-1/2}$ and $p |N|^{-1/2}$. Thus, at points not in S , n_μ can be normalized and the usual conservation equation in Σ^L and Σ^E is recovered [20]. It must be stressed here that ϱ and p in (5)-(6) are naturally defined as eigenvalues of $\tau_{\mu\nu}$, without invoking ad hoc assumptions, in contrast with the definition of ϱ given in earlier works [12] which has opposite sign in Σ^E .

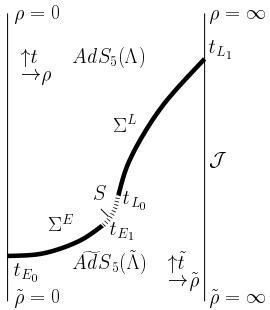


FIG. 1. Conformal diagram for AdS_5 and \widetilde{AdS}_5 matched along the changing signature hypersurface Σ , represented by the curve going from t_{E_0} to t_{L_1} . The bulk is the product of this diagram with S^3 . As usual, null lines are at 45° . The umbilical regions $t_{E_0} < t < t_{E_1}$ in Σ^E and $t_{L_0} < t < t_{L_1}$ in Σ^L are denoted by solid lines, whereas the non-umbilical intermediate region is denoted by a dotted line.

The global bulk thus defined is then constituted by two different regions of AdS_5 -type, separated by a brane which, if desired, can change signature. Timelike branes separating two AdS_5 bulks with different cosmological constants have been already studied, [16], and are included in our treatment. The two AdS_5 regions may be

interpreted as two fundamental states with different vacuum energies which can live together precisely due to the existence of the brane. Models describing jumps of the cosmological constant have been presented in different contexts, mainly in order to explain its small present day value (see [21] and references therein). In the case of signature-changing Σ , if the Lorentzian part of the brane is connected one can easily see that there must always exist a time t_{E_0} in the coordinates we are using such that the bulk is \widetilde{AdS}_5 for all times before t_{E_0} , see Fig. 1. Thus, we can think of \widetilde{AdS}_5 as the original bulk, which may represent a false vacuum. This vacuum would undergo a phase transition (similar to that of standard inflation, for instance) which occurs, as usual, in an acausal way, so that it can be modeled with the spacelike part Σ^E . In a region near S , and due probably to the matter fields present and their properties, Σ^E undergoes an internal process of signature change and it smoothly changes to the Lorentzian part Σ^L . Then, there is an epoch (up to the time t_{L_1} , see Fig. 1) in which the two vacua co-live separated in fact by a timelike brane. This part Σ^L of the brane would be our 4-dimensional world. Eventually, for all times after t_{L_1} , the bulk becomes AdS_5 with the constant Λ . Notice further that both the bulk and the brane are regular everywhere. In the brane, there appears a very distinguished instant of time, given by S , and a transition region around S (one part belonging to Σ^E and another part to Σ^L), which are quite remarkable from the inner point of view of the brane. They would correspond to the big-bang ‘singularity’, to the pre-big-bang Euclidean phase, and to the very early universe (possibly with an inflationary era), respectively.

Of course, any phase transition takes some (very small but finite) time, and thus the hypersurface description used here for Σ^E is an effective one. In our opinion, this is yet another positive property of our proposal, because it makes the explicit models *theoretically testable*, in the following sense. There must be a relation between the thickness of Σ^L —which is its *spatial extension*—, and that of Σ^E —which is its *temporal duration*—being both part of the same brane. For instance, in some brane scenarios [1] the thickness of Σ^L is of the order of the electroweak scale $m_{EW} \sim 1\text{TeV} \sim 10^{-16}\text{mm}$. This gives an estimation for the thickness of Σ^E of around $10^{-28} - 10^{-29}\text{s}$ and this should be in agreement with the time scales for the phase transitions in any microscopic proposal to describe the decay from $\tilde{\Lambda}$ to Λ . Obviously a similar restriction would happen if a different brane model, and hence a different thickness of Σ^L , is considered. Other thickness estimations would come from the length scales of graviton trapping [3], given by $(-6/\Lambda)^{1/2}$ [4], and also either from distances between branes or radius of compact extra dimensions ($< 1\text{mm}$ for experimental reasons).

There are very many possibilities to construct explicit models of the type we are considering. The free function

$a(\xi)$ can be determined once the matter on the brane is chosen. For simplicity, let us consider the case of a scalar field, assumed to have an unstable constant value in most of Σ^E , and to finally settle down to another stable constant value in most of Σ^L . Thus, the brane has a cosmological constant type energy-momentum in some large portions of both Σ^E and Σ^L . Only around the signature-changing set S the scalar field becomes dynamical. It is easy to see from (5)-(7) that $\tau_{\mu\nu}$ will take the form of a cosmological constant energy-momentum tensor if $\dot{a}^2 - N[a^2\alpha^2\lambda^2/(a^2 - 1) + 1] = 0$, where $\alpha > 0$ is a constant. Its general solution leads to the following implicit form of Σ :

$$F(t, \rho) = \alpha \sin \{\lambda(t - t_{L_1})\} \sqrt{1 + \lambda^2 \rho^2} - 1 = 0 \quad (8)$$

where t_{L_1} is a constant. The family (8) corresponds to the spherically symmetric umbilical hypersurfaces in AdS_5 , and their scalar curvature is given by ${}^{(4)}R = 12\lambda^2\alpha^2/(1 - \alpha^2)$. These Σ are spacelike for $\alpha > 1$, timelike for $0 < \alpha < 1$ and null for $\alpha = 1$. From our assumptions, the brane will be umbilical everywhere except for a region around S . Notice, however, that this transition region can be made *as small as desired*. We choose to describe the entire brane by keeping the functional form (8) and letting α become a function of ξ , which can be taken as any smooth function of t ($t > 0$) in the interval (t_{E_0}, t_{L_1}) with the following properties: $\alpha = \alpha_1 > 1$ for $t_{E_0} < t < t_{E_1}$, $\alpha = \alpha_2 < 1$ for $t_{E_1} < t_{L_0} < t < t_{L_1}$ and, in the intermediate region $t_{E_1} < t < t_{L_0}$, $\alpha(t)$ is an interpolating function between the two constants α_1 and α_2 . Observe that $t_{E_1} < t_{E_0} + \pi/(2\lambda)$ and $t_{L_0} > t_{L_1} - \pi/(2\lambda)$, so that the only requirement is that $\alpha_1^2 + \mu(1 - \alpha_1^2) > 0$, where we have set $\mu \equiv \tilde{\lambda}/\lambda$. The change of signature must necessarily happen in the transition region $t_{E_1} < t < t_{L_0}$.

The form of the brane as seen from \widetilde{AdS}_5 can be found from (4). It can be easily proven that the hypersurface in \widetilde{AdS}_5 is also umbilical in Σ^E and Σ^L . Hence, it must take the form (8) where $\rho, t, t_{L_1} \rightarrow \tilde{\rho}, \tilde{t}, \tilde{t}_{L_1}$. The corresponding constant $\tilde{\alpha}$ can be found from the matching conditions to be $\alpha/\sqrt{\alpha^2 + \mu^2(1 - \alpha^2)}$. Since the hypersurface is nowhere null in the umbilical regions, we can take a unit n_μ in order to define the distribution δ at Σ^E and Σ^L . With this choice, $\tau_{\mu\nu}$ takes the form

$$\begin{aligned} \tau_{\mu\nu}|_{(t_{E_0}, t_{E_1})} &= \frac{3\lambda}{\sqrt{\alpha_1^2 - 1}} [\sqrt{\mu^2(1 - \alpha_1^2) + \alpha_1^2} - 1] \\ &\quad \times (g_{\mu\nu} + n_\mu n_\nu), \\ \tau_{\mu\nu}|_{(t_{L_0}, t_{L_1})} &= \frac{3\lambda}{\sqrt{1 - \alpha_2^2}} [\sqrt{\mu^2(1 - \alpha_2^2) + \alpha_2^2} - 1] \\ &\quad \times (g_{\mu\nu} - n_\mu n_\nu). \end{aligned} \quad (9)$$

These expressions show that the tension on the umbilical region of Σ^L is positive if and only if $\tilde{\lambda} < \lambda$. This

has a nice physical interpretation because the energy-density of the original bulk \widetilde{AdS}_5 is less negative than the energy-density of the final bulk AdS_5 , in accordance with the possibility that AdS_5 is more stable than \widetilde{AdS}_5 . Furthermore, we find from (9) that, when $\tilde{\lambda} < \lambda$, the energy-density on the umbilical part of Σ^E measured by *any* timelike observer is also positive. Again this is physically reasonable.

Of course, many other possibilities are allowed and, for any type of energy-momentum tensor, (5) can be solved to get Σ . Hence, any closed RW brane can be modeled by our construction from times not too close to S .

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